Weighted Null Space Fitting
A link between the
Prediction Error Method and Subspace Identification

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Outline

Problem Setting

True System:

$$y_t = \frac{L_\circ(q)}{F_\circ(q)} u_t + \frac{C_\circ(q)}{D_\circ(q)} e_t$$

Model:

$$y_t = \frac{L(q,\theta)}{F(q,\theta)} u_t + v_t$$

$$v_t = H(q,\theta) e_t$$

$$H(q,\theta) = \frac{C(q,\theta)}{D(q,\theta)}$$

Estimate θ!

Prediction Error Method

Minimize cost function:

$$V_N^{\mathsf{PEM}}(\theta) = \sum_{t=1}^{N} \left[\frac{1}{H(q,\theta)} \left(y_t - \frac{L(q,\theta)}{F(q,\theta)} u_t \right) \right]^2$$

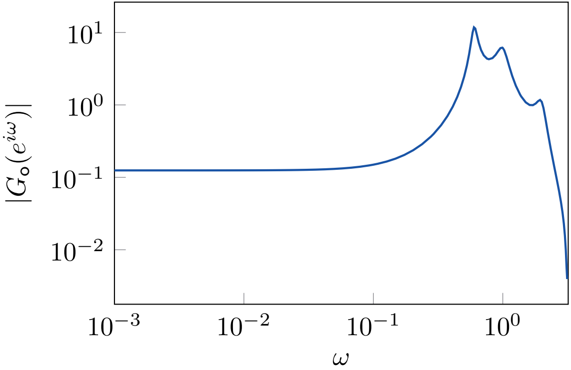
Non-convex!

Test example

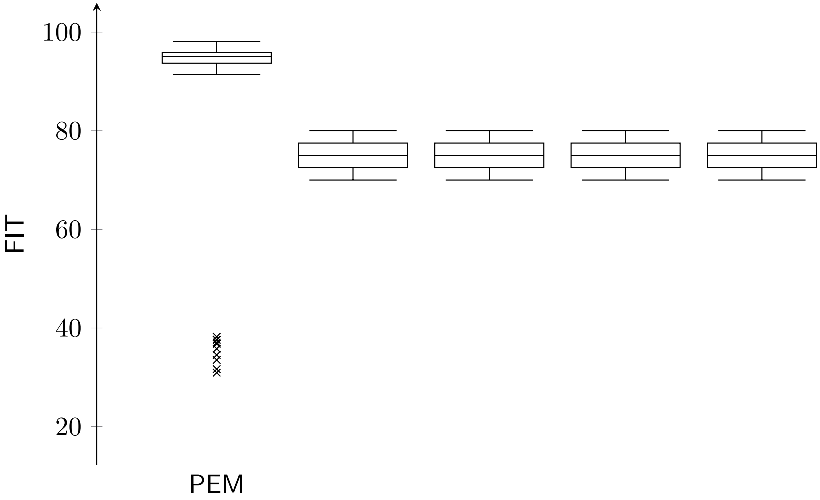
Box-Jenkins:

$$y_t = \frac{L_\circ(q)}{F_\circ(q)} u_t + \frac{1 + 0.8q^{-1}}{1 - 0.9q^{-1}} e_t$$

System (6^{th} order):



Simulation



Simulation

Alternatives and complements to PEM:

- Instrumental variable methods
- Subspace methods
- Iterative least-squares methods
- Multi-step high-order least-squares methods

Iterative Least Squares Methods

Output Error Models: $y_t = \frac{L(q)}{F(q)} u_t + e_t$

PEM: $V_N^{\text{PEM}}(\theta) = \sum_{t=1}^{N} (y_t - \frac{L(q)}{F(q)} u_t)^2$

Non-convex :(Tempting to try:

$$\sum_{t=1}^{N} (F(q) y_t - L(q) u_t)^2$$

Modified PEM is Least-Squares!
but biased since we (for open loop data) are minimizing

$$\sum_{t=1}^{N} \left(F(q) \left(\frac{L_o(q)}{F_o(q)} u_t + e_t \right) - L(q) u_t \right)^2 \approx \sum_{t=1}^{N} \left(\frac{F(q) L_o(q) - F_o(q) L(q)}{F_o(q)} u_t \right)^2 + \sum_{t=1}^{N} (F(q) e_t)^2$$

Steiglitz-McBride

Step 1: Estimate $F(q,\theta)y_t = L(q,\theta)u_t + e_t \quad \Longrightarrow \quad \hat{\theta}_N^1$

Step 2: Filter the data according to

$$y_t^f = \frac{1}{F(q,\hat{\theta}_N^1)} y_t, \qquad u_t^f = \frac{1}{F(q,\hat{\theta}_N^1)} u_t$$

Step 3: Estimate $F(q,\theta)y_t^f = L(q,\theta)u_t^f + e_t \quad \Longrightarrow \quad \hat{\theta}_N^2$

Iterate!

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\left[\frac{F(q,\theta)}{F(q,\hat{\theta}_N^k)} y_t - \frac{L(q,\theta)}{F(q,\hat{\theta}_N^k)} u_t\right]^2$$

$\theta_N^k \to \theta_o$ as $k \to \infty$ and $N \to \infty$

...but the noise must be white and $\hat{\theta}_N$ is not asymptotically efficient!

Multi-Step High Order Methods

- Prefiltering
- Residual estimation
- Optimal model reduction
- Weighted Null Space Fitting (WNSF)

High-Order ARX Models

$$y_t = G_\circ(q)u_t + H_\circ(q)e_t \iff A_\circ(q)y_t = B_\circ(q)u_t + e_t$$

$$A_\circ(q) = \frac{1}{H_\circ(q)} = 1 + a_1^\circ q^{-1} + a_2^\circ q^{-2} + \cdots$$

$$B_\circ(q) = \frac{G_\circ(q)}{H_\circ(q)} = b_1^\circ q^{-1} + b_2^\circ q^{-2} + \cdots$$

$$A(q,\eta^n)y_t = B(q,\eta^n)u_t + e_t$$

$$A(q,\eta^n) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n} \qquad B(q,\eta^n) = b_1 q^{-1} + \cdots + b_n q^{-n}$$

$$\eta^n = \begin{bmatrix} a_1 & \cdots & a_n & b_1 & \cdots & b_n \end{bmatrix}^\top$$

Choose n “sufficiently large” for the truncation error to be “sufficiently small!”

Special case: High-Order FIR

$$y_t = B(q, \eta^n)u_t + e_t = G(q, \eta^n)u_t + e_t$$

Matrix form:

$$Y = T_{N\times n}(u)\eta + E, \quad T_{N\times n}(u) = \begin{bmatrix} u_1 & 0 & 0 & \dots & 0 \\ u_2 & u_1 & 0 & \dots & 0 \\ u_3 & u_2 & u_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_N & u_{N-1} & u_{N-2} & \dots & u_{N-n+1} \end{bmatrix}$$

$$\text{Cov } \hat{\eta}^n = \sigma^2 \left(T_{N\times n}^T(u) T_{N\times n}(u) \right)^{-1}$$

Prefiltering

$$y_t = \frac{L_o(q)}{F_o(q)} u_t + H_o(q) e_t, \quad \hat{A}(q) \approx H_o^{-1} \Rightarrow$$

$$\hat{A}(q) y_t = \frac{L_o(q)}{F_o(q)} \hat{A}(q) u_t + \hat{A}(q) H_o(q) e_t \approx \frac{L_o(q)}{F_o(q)} \hat{A}(q) u_t + e_t$$

Now use SM on prefiltered data $\{\hat{A}(q) y_t, \hat{A}(q) u_t\}$

The Box-Jenkins Steiglitz McBride Method.

For open loop data, BJSM is

- consistent
- asymptotically efficient for Gaussian noise (even for OE models!)
- still need to iterate ($k \to \infty$)

Prefiltering

Notice BJSM uses both high-order model and data when estimating the model.

However, the high-order ARX model is (almost) a sufficient statistic, so from a statistical perspective we should be able to use this model only when estimating the model.

Residual Estimation

For a while we will for simplicity consider open loop data and the OE-case:

$$y_t = \frac{L_o(q)}{F_o(q)} u_t + e_t$$

and use a high-order FIR model

$$y_t = G(q) u_t + e_t, \; G(q) = \sum_{k=1}^{n} \eta_k q^{-k}$$

- High order predictor: $\hat{y}_t = \hat{G}(q) u_t$
- Form residuals: $\varepsilon_t = y_t - \hat{y}_t$
- Use residuals in estimation: $y_t = \frac{L(q)}{F(q)} u_t + \varepsilon_t$

Net result:

$$y_t = \frac{L(q)}{F(q)} u_t + y_t - \hat{y}_t \;\Rightarrow\; \hat{y}_t = \frac{L(q)}{F(q)} u_t$$

$V_N^{\mathsf{RE}}(\theta) = \sum_t \left(\hat{y}_t - \frac{L(q)}{F(q)} u_t \right)^2$

Simulated output used instead of the real output - only high order model used!

Optimal model reduction

Model reduction taking the statistical properties of the high order estimate into account.

- Use the (asymptotic) distribution of $\hat{\eta}$

$$V_N^{\text{E-ML}}(\theta) = \left(\eta^n(\theta) - \hat{\eta}_N^n\right)^\top \text{cov}[\hat{\eta}_N^n]^{-1} \left(\eta^n(\theta) - \hat{\eta}_N^n\right)$$

The Extended Invariance Principle (EXIP)

- Use the asymptotic distribution of $G(e^{i\omega}, \hat{\eta}_N^n)$

$$\sqrt{N}(G(e^{i\omega}, \hat{\eta}_N^n) - G_o(e^{i\omega}))) \sim AsN\left(0, \frac{\sigma^2}{\Phi_u(\omega)}\right)$$

$$V_N^{\text{A-ML}}(\theta) = \int_0^{2\pi} |G(e^{i\omega}, \hat{\eta}_N^n) - G(e^{i\omega}, \theta)|^2 \Phi_u(\omega) d\omega$$

Asymptotic ML

Weighted Null Space Fitting

$$\frac{L(q)}{F(q)} = G(q) \quad \Rightarrow \quad F(q)G(q) - L(q) = 0$$

$$\Rightarrow \quad F(q)\hat{G}(q) - L(q) = F(q)(G(q) + \Delta_G(q)) - L(q) = F(q)\Delta_G(q)$$

Find F and L s.t. $F(q)\hat{G}(q) - L(q)$ behaves in a statistical way as $F(q)\Delta_G(q)$

In equation form: $F(q) = 1 + f_1 q^{-1} + \ldots f_m q^{-m} = 1 + \tilde{F}(q) \Rightarrow$

$$F(q)G(q) - L(q) = (1 + \tilde{F}(q))G(q) - L(q) = G(q) - \begin{bmatrix} 1 & -G(q) \end{bmatrix} \begin{bmatrix} L(q) \\ \tilde{F}(q) \end{bmatrix}$$

$$= g_1 q^{-1} + \ldots g_n q^{-n} - \begin{bmatrix} 1 & -(g_1 q^{-1} + \ldots + g_n q^{-n}) \end{bmatrix} \begin{bmatrix} l_1 q^{-1} + l_m q^{-m} \\ f_1 q^{-1} + \ldots + f_m q^{-m} \end{bmatrix}$$

$$\Leftrightarrow \ \eta - Q(\eta)\theta$$

$$V_N^{\text{WNSF}}(\theta) = (\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)^\top \left(T_{n\times n}(F(q,\theta)) \text{cov}[\hat{\eta}_N^n] T_{n\times n}^T(F(q,\theta)) \right)^{-1} (\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)$$

$T_{n\times n}(F(q,\theta))$ $n \times n$ lower Toeplitz matrix of coefficients of $F(q,\theta)$

Asymptotic Equivalence

Residual estimation, optimal model order reduction and WNSF are equivalent for large sample sizes.

"Proof"

- Residual estimation: $V_N^{\text{RE}}(\theta) = \sum_t \left(\hat{y}_t - \frac{L(q)}{F(q)} u_t\right)^2$
- Asymptotic ML: $V_N^{\text{A-ML}}(\theta) = \int_0^{2\pi} |G(e^{i\omega}, \hat{\eta}_N^n) - G(e^{i\omega}, \theta)|^2 \Phi_u(\omega) d\omega \approx$ $\int_0^{2\pi} |G(e^{i\omega}, \hat{\eta}_N^n) - G(e^{i\omega}, \theta)|^2 |U_N(e^{i\omega})|^2 d\omega =$ (Parseval) $= \sum_t \left(\hat{y}_t - \frac{L(q)}{F(q)} u_t\right)^2$
- EXIP: $V_N^{\text{E-ML}}(\theta) = \left(\hat{\eta}^n(\theta) - \hat{\eta}_N^n\right)^\top \text{cov}[\hat{\eta}_N^n]^{-1} \left(\eta^n(\theta) - \hat{\eta}_N^n\right)$
  but $\sigma^2 \text{cov}[\hat{\eta}_N^n]^{-1} = T_{N\times n}(u)^T T_{N\times n}(u)$, and

$$T_{N\times n}(u)(\eta^n(\theta) - \hat{\eta}_N^n) = \begin{bmatrix} G(q,\theta)u_1 - G(q,\hat{\eta}_N^n)u_1 \\ \vdots \\ G(q,\theta)u_N - G(q,\hat{\eta}_N^n)u_N \end{bmatrix} = \begin{bmatrix} G(q,\theta)u_1 - \hat{y}_1 \\ \vdots \\ G(q,\theta)u_N - \hat{y}_N \end{bmatrix}$$

- WNSF:
  $V_N^{\text{WNSF}}(\theta) = (\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)^\top \left(T_{n\times n}(F(q,\theta)) \text{cov}[\hat{\eta}_N^n] T_{n\times n}^T(F(q,\theta))\right)^{-1} (\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)$
  $\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta \Leftrightarrow$

$$\begin{aligned} F(q,\theta)G(q,\hat{\eta}_N^n) - L(q,\theta) &= F(q,\theta)G(q,\hat{\eta}_N^n) - L(q,\theta) - \underbrace{(F(q,\theta)G(q,\eta(\theta)) - L(q,\theta))}_{0} \\ &= F(q,\theta)(G(q,\hat{\eta}_N^n) - G(q,\eta(\theta))) \Leftrightarrow T_{n\times n}(F(q,\theta))(\hat{\eta}_N^n - \eta(\theta)) \end{aligned}$$

Towards Least-Squares

Residual Estimation: $\sum_t (\hat{y}_t - \frac{L(q)}{F(q)} u_t)^2$

Still as non-convex as PEM. Advantage??

Modified cost function: $\sum_t (F(q)\hat{y}_t - L(q)u_t)^2$

Least-Squares! but is it any good, c.f. modified PEM?

Let the order n of the FIR model G grow to infinity: $n(N) \to \infty$ as $N \to \infty$.
$\Rightarrow \hat{G}(q) \to G_o(q) \Rightarrow$ Error in $\hat{y}_t$ vanishes as $N \to \infty \Rightarrow \hat{y}_t = G_o(q)u_t$

Least-squares estimate consistent (unlike modified PEM)!

Consistent Least-Squares Estimation

- Residual Estimation: $\sum_t (F(q)\hat{y}_t - L(q)u_t)^2$
- Optimal model order reduction (Asymptotic ML): $\int_0^{2\pi} |F(e^{i\omega})G(e^{i\omega}, \hat{\eta}_N^n) - L(e^{i\omega})|^2 \Phi_u(\omega) d\omega$
- WNSF: $(\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)^\top \left(T_{n\times n}(1)\text{cov}[\hat{\eta}_N^n]T_{n\times n}^T(1)\right)^{-1} (\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)$

All consistent if $n(N) \to \infty$ at a suitable rate:

- Not too slow: $n(N)/(\log(N))^{1+\delta} \to \infty$ for some $\delta > 0$
- Not too fast: $n^{4+\delta}(N)/N \to 0$

Towards Asymptotically Efficient Least-Squares

Residual Estimation: $\sum_t (F(q)\hat{y}_t - L(q)u_t)^2$

$$\hat{y}_t = G(q, \hat{\eta}_N^n)u_t = G_o(q)u_t + \Delta_G(q)u_t$$

$$F(q)\hat{y}_t - L(q)u_t = (F(q)G_o(q) - L(q))u_t + F(q)\Delta_G(q)u_t$$

$F(q)\Delta_G(q)u_t$ random error term. Has to be white for asymptotic efficiency.

$\Delta_G(q) \Leftrightarrow \hat{\eta}_N^n - \eta_o$ which has covariance $\sigma^2 \left(T_{N\times n}^T(u) T_{N\times n}(u)\right)^{-1}$

$\Rightarrow \Delta_G(q)u_t$ is temporally white!

$\Rightarrow F(q)\Delta_G(q)u_t$ is NOT temporally white :(

Asymptotically Efficient Least-Squares

Idea: Two-steps

Residual Estimation:

1. Minimize $\sum_t (F(q)\hat{y}_t - L(q)u_t)^2 \quad \Rightarrow \hat{L},\ \hat{F}$ (consistent)
2. $\sum_t (F(q)\frac{1}{\hat{F}(q)}\hat{y}_t - L(q)\frac{1}{\hat{F}(q)}u_t)^2 \quad \Rightarrow \hat{\hat{L}},\ \hat{\hat{F}}$

Result: $\hat{\hat{L}},\ \hat{\hat{F}}$ asymptotically efficient if $n(N) \to \infty$ at a suitable rate.

- Optimal model order reduction (Asymptotic ML):

$$\int_0^{2\pi} |F(e^{i\omega})G(e^{i\omega}, \hat{\eta}_N^n) - L(e^{i\omega})|^2 \frac{\Phi_u(\omega)}{|\hat{F}(e^{i\omega})|^2} d\omega$$

- WNSF: $(\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)^\top \left(T_{n\times n}(\hat{F}(q))\text{cov}[\hat{\eta}_N^n]T_{n\times n}^T(\hat{F}(q))\right)^{-1}(\hat{\eta}_N^n - Q(\hat{\eta}_N^n)\theta)$

Summary: Three steps: i) High order LS, ii) OLS, iii) WLS

A Brief History of Iterative Least-Squares and High-Order Methods

- Durbin 1959: MA. High order AR-model. Clever way of using that model as weighting as well $\Rightarrow$ Only two steps!
- Durbin 1960: ARMA. High order AR-model. Alternate between MA-part (using previous result), and AR-part (easy).
- Santathan & Koerner 1963: Steiglitz-McBride in frequency domain.
- Steiglitz-Mcbride 1965: Steiglitz-McBride iterations.
- Mayne & Firoozan 1982: ARMA. Residual estimation. All three steps. Consistency & asymptotic efficiency but when first N and then n tends to infinity.
- Hannan & Rissanen 1982: ARMA. Residual estimation. Uses model in step 2 to form new residual estimate. Order estimation. Recursive. $n = n(N)$. Consistency and asymptotic efficiency.
- Hannan & Kavaleris 1983: As Mayne & Firoozan but consistency analyzed for $n = n(N)$.
- Mayne, Åström & Clark 1984: As Mayne & Firoozan but recursive.
- Hannan & Kavaleris 1984: As Hannan & Rissanen but multivariate & order recursive.

A Brief History of Iterative Least-Squares and High-Order Methods

- Zhu 1989: ASYM - Asymptotic ML in time-domain (=Residual estimation)
- Wahlberg 1989: Asymptotic ML.
- Zhu 2011: Box-Jenkins Steiglitz-Mcbride. Prefiltering method.
- Dufour & Jouini 2014: VARMA. Multi-step.
- Galrinho, Rojas and Hjalmarsson 2014: WNSF.
- Everitt, Galrinho and Hjalmarsson 2017: MORSM. Residual estimation.
- Fang, Galrinho & Hjalmarsson 2017: WNSF. Recursive.

Multi-step LS vs subspace identification

Subspace id:

1) Estimate Hankel matrix

$$\mathcal{H} = \begin{bmatrix} g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ g_3 & g_4 & g_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{O}_e \mathcal{C}_e$$

2) Obtain estimate of extended observability matrix $\mathcal{O}_e$ using SVD

$$\begin{aligned} W_1 \hat{\mathcal{H}} W_2 &= U S V^T \\ \mathcal{O}_e &= W_1^{-1} \bar{U} \bar{S}^{1/2} \end{aligned}$$

where $\bar{U}$ and $\bar{S}$ truncated versions of U and S.

This means that the range space of $\mathcal{H}$ is estimated.

3) Estimate state-space matrices from $\mathcal{O}_e$.

Data only used in Step 1. W_1 can be used to affect the statistical accuracy. Not clear what the optimal weighting is.

Multi-step LS vs subspace identification

WNSF:

$$F(q)G(q) - L(q) = 0 \;\Rightarrow\; (1+f_1q^{-1}+\ldots+f_mq^{-m})(g_1q^{-1}+\ldots g_nq^{-n}) - (l_1q^{-1}+\ldots+l_mq^{-m}) = 0$$

Look at delays higher than m:

$$\begin{bmatrix} g_1 & g_2 & g_3 & \cdots & g_f \\ g_2 & g_3 & g_4 & \cdots & g_{f+1} \\ g_3 & g_4 & g_5 & \cdots & g_{f+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_f & g_{f+1} & g_{f+2} & \cdots & g_{2f-1} \end{bmatrix} \begin{bmatrix} f_m \\ \vdots \\ f_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

Multi-step LS vs subspace identification

but also

$$\begin{bmatrix} g_1 & g_2 & g_3 & \cdots & g_f \\ g_2 & g_3 & g_4 & \cdots & g_{f+1} \\ g_3 & g_4 & g_5 & \cdots & g_{f+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_f & g_{f+1} & g_{f+2} & \cdots & g_{2f-1} \end{bmatrix} \begin{bmatrix} f_m & 0 & 0 & \ldots & 0 \\ f_{m-1} & f_m & 0 & \ldots & 0 \\ f_{m-2} & f_{m-1} & f_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ f_1 & f_2 & f_3 & \ldots & 0 \\ 1 & f_1 & f_2 & \ldots & f_m \\ 0 & 1 & f_1 & \ldots & f_{m-1} \\ 0 & 0 & 1 & \ldots & f_{m-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} = 0$$

- $\mathcal{H}$ has rank $m \Rightarrow$ Nullspace of $\mathcal{H}$ has dim $f - m$
- Right hand factor has $f - m$ columns
- Right hand factor has full column rank
- Parametrization of null-space of $\mathcal{H}$!

Multi-step LS vs subspace identification

Estimate $f_1, \dots, f_m$ by solving

$$\begin{bmatrix} \hat{g}_1 & \hat{g}_2 & \hat{g}_3 & \dots & \hat{g}_f \\ \hat{g}_2 & \hat{g}_3 & \hat{g}_4 & \dots & \hat{g}_{f+1} \\ \hat{g}_3 & \hat{g}_4 & \hat{g}_5 & \dots & \hat{g}_{f+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{g}_f & \hat{g}_{f+1} & \hat{g}_{f+2} & \dots & \hat{g}_{2f-1} \end{bmatrix} \begin{bmatrix} f_m & 0 & 0 & \dots & 0 \\ f_{m-1} & f_m & 0 & \dots & 0 \\ f_{m-2} & f_{m-1} & f_m & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ f_1 & f_2 & f_3 & \dots & 0 \\ 1 & f_1 & f_2 & \dots & f_m \\ 0 & 1 & f_1 & \dots & f_{m-1} \\ 0 & 0 & 1 & \dots & f_{m-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = 0$$

$\hat{g}$ noisy. $\Rightarrow$ Need to take statistics into account (c.f. subspace id)

Same problem as in subspace id? No!!!

In this case we can vectorize the system of equations $\Rightarrow$ WNSF!

Simpler to in a statistically efficient way estimate elements in the null-space than elements in the range space of a matrix

Multi-step LS vs subspace identification

Summary:

Method	Subspace id	Multi-step LS
Subspace	Range space	Null space
Weighting of	$\hat{\mathcal{H}}$	vec $\left\{\hat{\mathcal{H}}\right\}$
Estimation method	SVD+LS	LS
Can incorporate structural information	NO	YES
Consistency	YES	YES
Asymptotic efficiency	Special cases	YES

MIMO models

Matrix-Fraction Description (MFD) OE-MIMO:

$$y_t = F^{-1}(q)L(q)u_t + e_t$$

High order model: MIMO-FIR

$$y_t = G(q)u_t + e_t$$

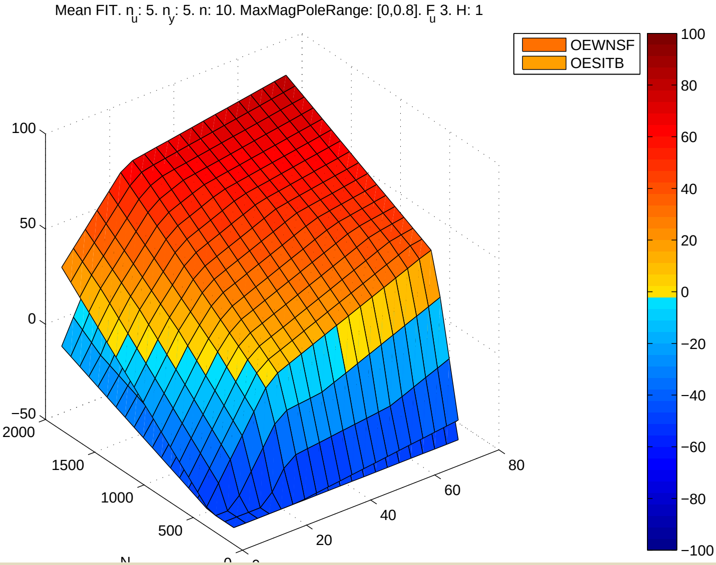
$$F^{-1}(q)L(q) = G(q) \;\Leftrightarrow F(q)G(q) - L(q) = 0$$

Same as in the SISO case!

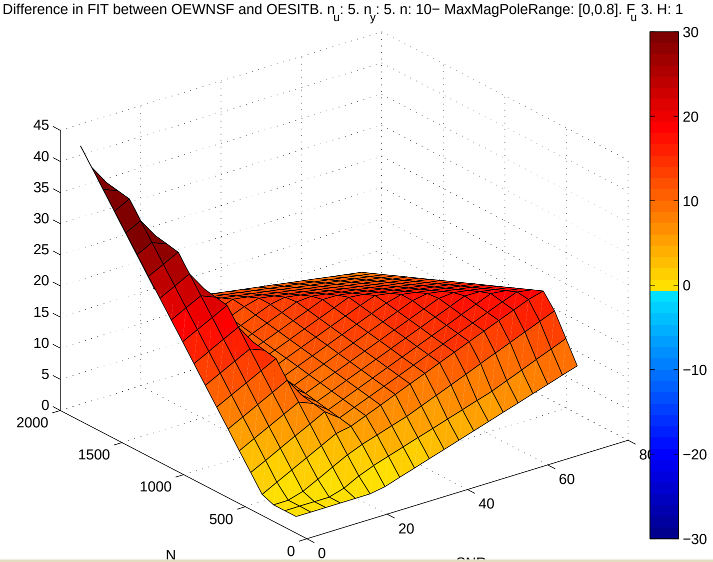
OE, ARMAX, BJ, MAX, ...

MFD, element-wise parameterizations

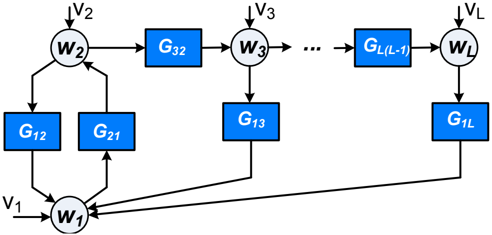
MIMO models: Output error with element wise parametrization



MIMO models: Output error with element wise parametrization



Dynamic Network Identification



$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t)$$

Interconnection structure given by

$$G(q) = \begin{bmatrix} 0 & G_{12}(q) & G_{13}(q) \\ G_{21}(q) & 0 & G_{23}(q) \\ G_{31}(q) & G_{32}(q) & 0 \end{bmatrix}$$

Dynamic Network Identification

$$G(q) = \begin{bmatrix} 0 & G_{12}(q) & G_{13}(q) \\ G_{21}(q) & 0 & G_{23}(q) \\ G_{31}(q) & G_{32}(q) & 0 \end{bmatrix}$$

Suppose

$$G(q) = D^{-1}(q)N_G(q), \quad R(q) = D^{-1}(q)N_R(q), \quad H(q) = D^{-1}(q)N_H(q)$$

$$w = Gw + Rr + He \;\Leftrightarrow\; D(q)w(t) = N_G(q)w(t) + N_R(q)r(t) + N_H e(t)$$

which can be written

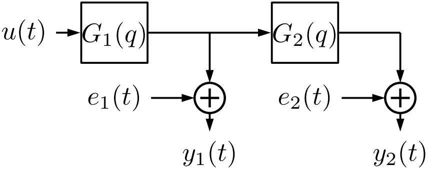
$$(D(q) - N_G(q))w(t) = N_R(q)r(t) + N_H e(t)$$

ARMA!

A range of structures can be accomodated for. For example

$D(q)$ diagonal: All transfer functions to one node have the same poles.

Cascade Networks



$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{L_1(q,\theta)}{F_1(q,\theta)} \\ \frac{L_2(q,\theta)}{F_2(q,\theta)} \frac{L_1(q,\theta)}{F_1(q,\theta)} \end{bmatrix} u(t) + e(t)$$

PEM: can be difficult...

Cascade Networks

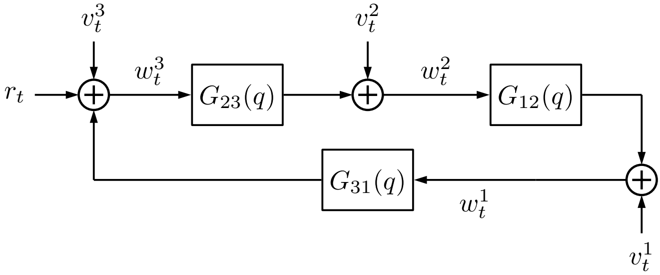
$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{L_1(q,\theta)}{F_1(q,\theta)} \\ \frac{L_2(q,\theta)}{F_2(q,\theta)} \frac{L_1(q,\theta)}{F_1(q,\theta)} \end{bmatrix} u(t) + e(t)$$

$$\approx \begin{bmatrix} \sum_{k=1}^{n} g_k^{(1)} q^{-k} \\ \sum_{k=1}^{n} g_k^{(21)} q^{-k} \end{bmatrix} u(t) + e(t) \qquad \Longrightarrow \qquad \hat{g}_k^{(1)}, \hat{g}_k^{(21)} \text{ (LS)}$$

$$\begin{cases} \frac{L_1(q,\theta)}{F_1(q,\theta)} = \bar{G}_1(q,g) \\ \frac{L_2(q,\theta)}{F_2(q,\theta)} \bar{G}_1(q,g) = \bar{G}_{21}(q,g) \end{cases} \Leftrightarrow \begin{cases} F_1(q,\theta)\bar{G}_1(q,g) - L_1(q,\theta) = 0 \\ F_2(q,\theta)\bar{G}_{21}(q,g) - L_2(q,\theta)\bar{G}_1(q,g) = 0 \end{cases}$$

WNSF can be applied with optimal asymptotic properties

Errors-in-Variables Problems in Dynamic Networks

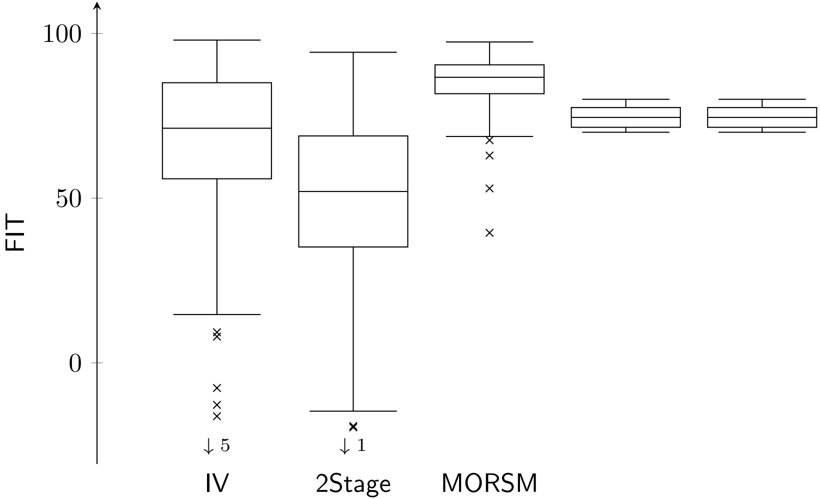


Measurements: $\tilde{w}_t = w_t + s_t$

Estimate G_{12}. Errors-in-variables problem!

- IV
- Two-stage methods
- WNSF, but requires more steps (no time for this, unfortunately)

Dynamic Networks: Simulation



Noise with High-order Dynamics

A low-order parametrization of the noise model...

- ...will not give asymptotic efficiency in open loop
- ...will not give consistency in closed loop

In Step 1, WNSF capture the noise with the non-parametric model

In Step 2, we may only compute a parametric model of the plant!

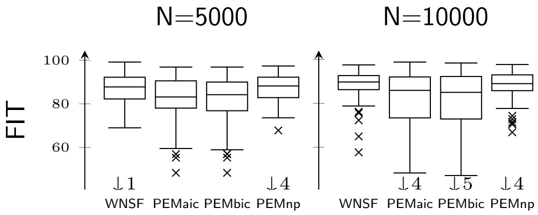
$$H(q,\theta) = \frac{1}{A(q,\eta)} \qquad G(q,\theta) = \frac{B(q,\eta)}{A(q,\eta)}$$

Noise with High-order Dynamics

True noise model given by very long FIR without a low-order parametrization.

Noise models:

- WNSF and PEMnp use a noise model $H(q,a) = 1/[1+\sum_{k=1}^{70} a_k q^{-k}]$.
- PEMaic and PEMbic use a noise model $H(q;c,d) = [1+\sum_{k=1}^{m} c_k q^{-k}]/[1+\sum_{k=1}^{m} d_k q^{-k}]$, $m = \{1, ..., 30\}$, with m decided with AIC/BIC.



Average computational times [s]

N	5000	10000
WNSF	0.907	1.29
$\text{PEM}_{\text{aic,bic}}$	26.8	38.1
PEM_{np}	133	236

Online Identification

Recursive PEM:

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{1}{t}\Lambda_t^{-1}\psi_t(\hat{\theta}_{t-1})\varepsilon_t(\hat{\theta}_{t-1})$$

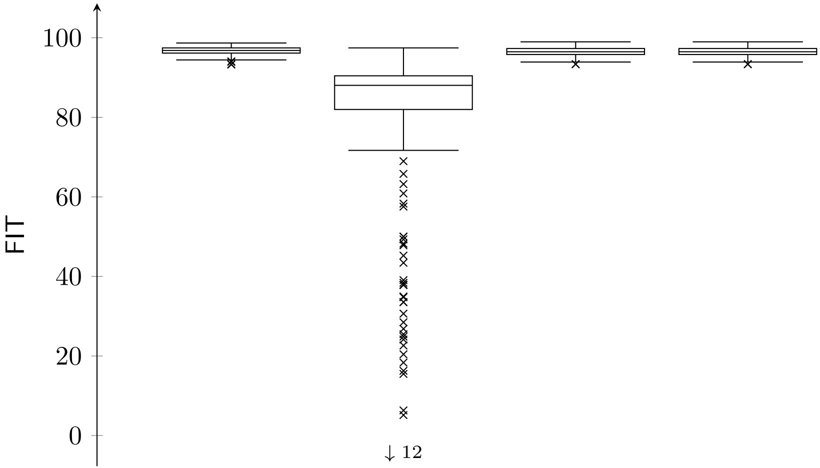
The gradient $\psi_t(\theta)$ and the prediction error $\varepsilon_t(\theta)$ cannot be computed with fixed-size memory $\Longrightarrow$ approximations!

Recursive WNSF:

- ARX model can be computed recursively
- Parametric estimate update identical to offline method

$$\hat{\theta}_t = \left[Q^\top(\hat{\eta}_t^n)W(\hat{\theta}_{t-1})Q(\hat{\eta}_t^n)\right]^{-1} Q^\top(\hat{\eta}_t^n)W(\hat{\theta}_{t-1})\hat{\eta}_t^n \quad (1)$$

Online Identification



Nonlinear Models

Rational in parameters models:

$$y(t) = \frac{f(\varphi(t))\theta}{1 + g(\varphi(t))\theta} + e(t)$$

where $\varphi(t)$ function of past inputs and outputs.

Multi-step LS procedure:

i) High order expansion: $y(t) = \sum_{k=0}^{\infty} \alpha_k \gamma_k(\varphi(t))$
for example Taylor expansion

ii) Truncation and LS-estimation: $\hat{\alpha} = \text{argmin}_\alpha \sum_{t=1}^{N} \left(y(t) - \sum_{k=1}^{n} \alpha_k \gamma_k(\varphi(t))\right)^2$

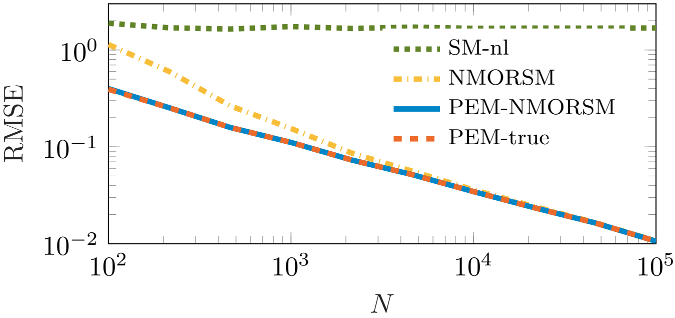
ii) Simulated model output: $\hat{y}(t) := \sum_{k=1}^{n} \hat{\alpha}_k \gamma_k(\varphi(t))$

iii) Residual estimation: $\hat{y}(t) \approx \frac{f(\varphi(t))\theta}{1+g(\varphi(t))\theta}$

iv) Multi-step LS: $\hat{\theta}_{k+1} = \text{argmin}_\theta \sum_{t=1}^{N} \left(\frac{\hat{y}(t)(1+g(\varphi(t))\theta) - f(\varphi(t))\theta}{1+g(\varphi(t))\hat{\theta}_k}\right)^2$

Nonlinear Models

$$y(t) = \frac{\theta_1 u(t-1)}{1+\theta_2 u^2(t)} + e(t) = \sum_{k=0}^{\infty} \alpha_k u(t-1) u^{2k}(t)$$



Conclusions

- PEM may require very accurate initial conditions to converge to the global minimum
  - Systems with several resonance peaks
  - Systems with widely spread eigenvalues
- Multi-step high order LS may be appropriate to handle these scenarios:
  - Less sensitive to the effect of the initial condition
  - Faster convergence
  - Asymptotically efficient
- Other scenarios where multi-step high-order LS may be useful:
  - MIMO
  - Online identification
  - Dynamic networks
  - Overparametrized models
  - Non-parametric noise spectra
  - Non-linear models